Pareto Front Approximation through a Multi-objective Augmented Lagrangian Method

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Single-point descent methods

- Extensions of classical scalar optimization techniques.
- Production of sequence of points asymptotically driven to optimality.
- `MultiObjectiveSteepestDescent` (`MOSD`) [Fliege and Svaiter, 2000]: one of the most famous.
- Drawback:
  - Single solution, no Pareto front approximation.
  - Pareto front useful to choose the most appropriate trade-off.

Our goals

- Focus on Augmented Lagrangian Algorithm for Multi-objective Optimization (`ALAMO`) [Cocchi and Lapucci, 2020]:
  - Single-point extension of scalar Augmented Lagrangian Method (`ALM`).
- Definition of an extended version of `ALAMO`:
  - Management of set of mutually non-dominated points;
  - Properties of convergence to Pareto-stationarity.

Problem

Definition

$$\min_{x \in \mathbb{R}^n} F(x) = (f_1(x), \ldots, f_m(x))^T$$
$$\text{s.t. } g(x) \leq 0 \qquad \text{(OPT)}$$

- $F : \mathbb{R}^n \to \mathbb{R}^m$
  - Continuously differentiable.
- $g : \mathbb{R}^n \to \mathbb{R}^p$
  - Continuously differentiable.
  - Component-wise convex.

Multi-objective Augmented Lagrangian

Definition

$$\mathcal{L}_\tau(x,\mu) = F(x) + \frac{\tau}{2}\left(\sum_{i=1}^{p}\left(\max\left\{0, g_i(x) + \frac{\mu_i}{\tau}\right\}\right)^2\right) e$$

- $\tau > 0 \in \mathbb{R}$: penalty parameter.
- $\mu \geq 0 \in \mathbb{R}^p$: Lagrange multipliers.

- Problem with `MO` Augmented Lagrangian.

$$\min_{x\in\mathbb{R}^n} \mathcal{L}_\tau(x,\mu) = (\ell_{\tau 1}(x,\mu), \ldots, \ell_{\tau m}(x,\mu))^T \qquad \text{(LAG)}$$

Pareto-stationarity

Definition

$\bar{x} \in \mathbb{R}^n$ *Pareto-stationary* for (LAG) if

$$\max_{j=1,\dots,m} \nabla \ell_{\tau j}(\bar{x}, \mu)^T d \geq 0 \ \forall d \in \mathbb{R}^n.$$

- Equivalent condition.

$$\min_{d \in \mathbb{R}^n} \max_{j=1,\dots,m} \nabla \ell_{\tau j}(\bar{x}, \mu)^T d = 0$$

- $\bar{x}$ not Pareto-stationary $\to \exists d \in \mathbb{R}^n$, descent for all objectives.

Steepest common descent direction

Definition

$$v(\bar{x}) = \arg\min_{\substack{d \in \mathbb{R}^n \\ \|d\| \leq 1}} \max_{j=1,\dots,m} \nabla \ell_{\tau j}(\bar{x}, \mu)^T d \qquad \text{(SC)}$$

- $\theta : \mathbb{R}^n \to \mathbb{R}$
  - $\theta(\bar{x})$: optimal value of (SC) at $\bar{x}$.

Steepest partial descent direction

Definition

Let $I \subseteq \{1, \ldots, m\}$ subset of objectives.

$$v^I(\bar{x}) = \underset{\substack{d \in \mathbb{R}^n \\ \|d\| \leq 1}}{\arg\min} \max_{j \in I} \nabla \ell_{\tau j}(\bar{x}, \mu)^T d \qquad \text{(SP)}$$

- $\theta^I(\bar{x})$: optimal value of (SP) at $\bar{x}$.

MOSD – Scheme

Input: $x^0 \in \mathbb{R}^n$.

While x^k not Pareto-stationary

- Compute
$$d^k \in \operatorname*{arg\,min}_{\substack{d \in \mathbb{R}^n \\ \|d\| \leq 1}} \max_{j=1,\dots,m} \nabla \ell_{\tau j}(x^k, \mu)^T d$$
- Find step size α_k (`ArmijoTypeLineSearch`)
- $x^{k+1} = x^k + \alpha_k d^k$

Output: x^k.

ArmijoTypeLineSearch – Scheme and properties

Inputs:

- $x, d \in \mathbb{R}^n$;
- $\alpha_0 > 0$;
- $\delta, \beta \in (0, 1)$.

While $\mathcal{L}_\tau(x + \alpha d, \mu) \not\leq \mathcal{L}_\tau(x, \mu) + \beta\alpha J_{\mathcal{L}_\tau}(x, \mu)d$

- $\alpha = \delta\alpha$

Output: α.

Finite termination

If $\mathcal{L}_\tau(\cdot, \mu)$ continuously differentiable and $J_{\mathcal{L}_\tau}(x, \mu)d < 0$ $(\theta(x) < 0)$, then $\exists\epsilon > 0$ such that

$$\mathcal{L}_\tau(x + td, \mu) < \mathcal{L}_\tau(x, \mu) + \beta t J_{\mathcal{L}_\tau}(x, \mu)d \quad \forall t \in (0, \epsilon].$$

ε-Pareto-stationarity

Definition

Let $\varepsilon \geq 0$. $\bar{x} \in \mathbb{R}^n$ ε-*Pareto-stationary* for (LAG) if

$$\min_{\substack{d \in \mathbb{R}^n \\ \|d\| \leq 1}} \max_{j=1,\ldots,m} \nabla \ell_{\tau j}(\bar{x}, \mu)^T d \geq -\varepsilon.$$

- ε-Pareto-stationarity $\rightarrow$ finite termination of `MOSD`.

FRONT-ALAMO – Scheme

Inputs:

- X^0, feasible non-dominated points for (OPT);
- $\{\varepsilon_k\} \subset \mathbb{R}$, decreasing sequence.

For $k = 0, 1, \ldots$

- Inner loop
  - Exploration and refinement phase for X^k
  - `MOSD` and `ArmijoTypeLineSearch`
  - Output $\rightarrow X^{k+1}$
- Update for multipliers and penalty parameter

FRONT-ALAMO – Inner loop

- $X^{k+1} = X^k$
- $\forall x_c \in X^k$, $\forall I \subseteq \{1, \dots, m\}$, if $\theta^I_k(x_c) < 0$
  - $d \in v^I_k(x_c)$
  - Find α (ArmijoTypeLineSearch) considering $\mathcal{L}^I_{\tau_k}(\cdot, \mu^k)$
  - Find z through MOSD with $\mathcal{L}_{\tau_k}(\cdot, \mu^k), x_c + \alpha d, \varepsilon_k$
  - If $\nexists y \in X^{k+1} : \mathcal{L}_{\tau_k}(y, \mu^k) \lneqq \mathcal{L}_{\tau_k}(z, \mu^k)$
    - $X^{k+1} = X^{k+1} \setminus \{x \in X^{k+1} \mid \mathcal{L}_{\tau_k}(z, \mu^k) \lneqq \mathcal{L}_{\tau_k}(x, \mu^k)\}$
    - $X^{k+1} = X^{k+1} \cup \{z\}$

FRONT-ALAMO – Update phase

- μ update
  - $\mu_i^{k+1} = \Pi_{[0,\bar{\mu}]}\left[\mu_i^k + \tau_k \max_{x\in X^{k+1}}\{g_i(x)\}\right] \quad \forall i = 1, \dots, p$
- τ update
  - $V_i^{k+1} = \min\left\{\min_{x\in X^{k+1}}\{-g_i(x)\}, \frac{\mu_i^k}{\tau_k}\right\} \qquad \forall i = 1, \dots, p$
  - $\left\|V^{k+1}\right\| > \sigma\left\|V^k\right\| \to \tau_{k+1} = \rho\tau_k$

FRONT-ALAMO – Properties – 1

- MOSD and line search finite termination $\rightarrow$ FRONT-ALAMO well-defined.

X^{k+1} features

$\forall k$, $\forall x^{k+1} \in X^{k+1}$:

- $\nexists y \in X^{k+1} : \mathcal{L}_{\tau_k}(y, \mu^k) \lneqq \mathcal{L}_{\tau_k}(x^{k+1}, \mu^k)$;
- x^{k+1} is ε_k-Pareto-stationary w.r.t. $\mathcal{L}_{\tau_k}(\cdot, \mu^k)$.

FRONT-ALAMO – Properties – 2

Let

- $\varepsilon_k \to 0$,
- $\{x^{k+1}\}$ such that $x^{k+1} \in X^{k+1}$.
- $K \subseteq \{0, 1, \dots\}$ and $\lim_{\substack{k\to\infty \\ k\in K}} x^{k+1} = \bar{x}$.

Limit points feasibility

$g(\bar{x}) \leq 0$.

Limit points Pareto-stationarity

$\bar{x}$ is Pareto-stationary for (OPT).

State-of-the-art algorithms

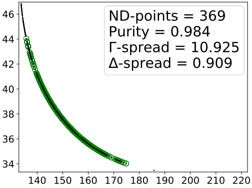
- MOSQP [Fliege and Vaz, 2016]: gradient-based method.
- DMS [Custódio et al., 2011]: derivative-free methodology.
- NSGA-II [Deb et al., 2002]: non-dominated sorting-based evolutionary algorithm.

Settings

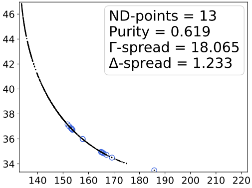
- Hyper-parameters values chosen by preliminary results or according to the algorithms authors.
- Code written in `Python3`.
- For each algorithm and problem, time limit of 2 minutes.
- For each problem, `NSGA-II` settings:
  - 10 runs;
  - 10 fronts compared based on *purity*;
  - Best front is chosen.

M-BNH2

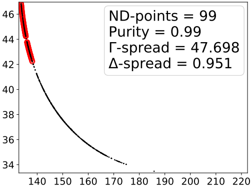
- $m = 2$; $n = 2$; 2 non-linear constraints.



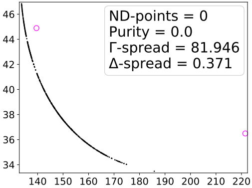
(a) `FRONT-ALAMO`



(b) `DMS`



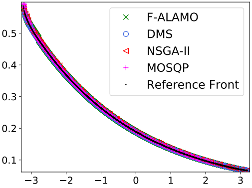
(c) `NSGA-II`



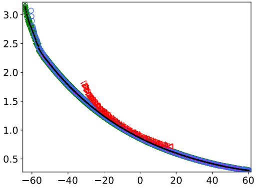
(d) `MOSQP`

LAP2

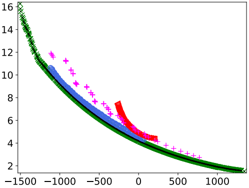
- $m = 2$; 1 non-linear constraint.



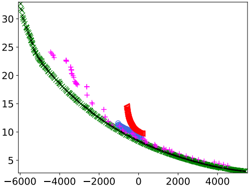
(a) LAP2, $n = 2$



(b) LAP2, $n = 10$



(c) LAP2, $n = 50$

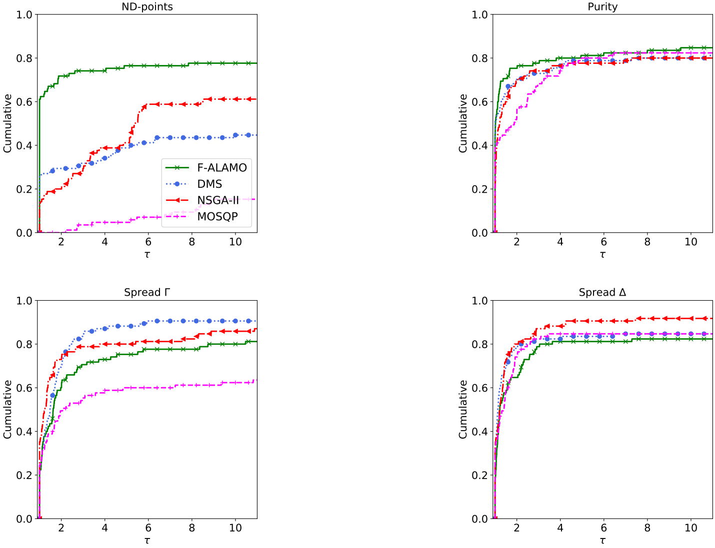


(d) LAP2, $n = 100$

Bound-constrained problems – 1

- CEC [Zhang et al., 2008].
  - $m = 2$: CEC1, CEC2, CEC3, CEC7.
  - $m = 3$: CEC8, CEC9, CEC10.
  - $n \in \{5, 10, 20, 30, 40, 50, 100, 200\}$.
- ZDT [Zitzler et al., 2000].
  - $m = 2$: ZDT1, ZDT2.
  - $n \in \{2, 5, 10, 20, 30, 40, 50, 100, 200\}$.
- MOP [Huband et al., 2006].
  - MOP1: $m = 2$, $n = 1$.
  - MOP2: $m = 2$, $n \in \{2, 5, 10, 20, 30, 40, 50, 100, 200\}$.
  - MOP3: $m = 2$, $n = 2$.

Bound-constrained problems – 2



Conclusions

- Smooth multi-objective optimization problems with convex constraints.
- Augmented Lagrangian method designed to produce Pareto front approximations.
- Properties of global convergence to Pareto-stationarity.
- Great performance w.r.t. state-of-the-art algorithms.

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