Sparse multiobjective optimization via concave approximations

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Sparse multiobjective problem

$$\begin{aligned} \min_x \quad & f_1(x), \ldots, f_{m-1}(x), \|x\|_0 \\ & x \in \mathcal{S} \end{aligned} \tag{P}$$

- $f_1(x), \ldots, f_{m-1}(x)$ continuously differentiable
- $\mathcal{S}$ convex and compact

Problem

$\|x\|_0$ is not continuous

Solution: concave approximations

Replace $\|x\|_0$ with a continuously differentiable concave function

$$\|x\|_0 = \sum_{i=1}^{n} s(|x_i|)$$

$$s : \mathbb{R} \to \mathbb{R}, \;\; s(t) = \begin{cases} 1, t > 0, \\ 0, t \leq 0. \end{cases}$$

Possible approximations

$$\|x\|_0 \approx \sum_{i=1}^{n} \left(1 - e^{-\alpha|x_i|}\right)$$

$$\|x\|_0 \approx \sum_{i=1}^{n} \log(\epsilon + |x_i|)$$

Approximated problem

$$\min_{x} \quad f_1(x), \ldots, f_{m-1}(x), \sum_{i=1}^{n} f^{\lambda}(|x_i|)$$

$$x \in \mathcal{S}$$

Approximated smooth problem

$$
\begin{aligned}
\min_{x,y} \quad & f_1(x), \ldots, f_{m-1}(x), \sum_{i=1}^{n} f^{\lambda}(y_i) \\
& x \in \mathcal{S} \\
& -y_i \leq x_i \leq y_i \quad \forall i = 1, \ldots, n
\end{aligned}
\qquad (\tilde{P}^{\lambda})
$$

Assumptions on f^λ

$\exists \bar{\lambda}$ such that $\forall \{\lambda_k\} \to \bar{\lambda}$:

- $\forall y_i > 0$, $f^{\lambda_k}(0) < f^{\lambda_k}(y_i)$ and

$$\lim_{k\to\infty} f^{\lambda_k}(0) < \lim_{k\to\infty} f^{\lambda_k}(y_i) < \infty;$$

- $\forall y_i \geq 0$: either

$$\lim_{k\to\infty} f^{\lambda_k}(y_i) = \begin{cases} 1 & \text{if } y_i > 0, \\ 0 & \text{if } y_i = 0, \end{cases}$$

or

$$\lim_{k\to\infty} f^{\lambda_k}(0) = -\infty.$$

Theorem:

- Let $\{\lambda_k\}$ such that
$$\lim_{k\to\infty} \lambda_k = \bar{\lambda};$$
- Let $\{\tilde{P}^{\lambda_k}\}$ such that $\forall k$
$$f_m(y) = \sum_{i=1}^{n} f^{\lambda_k}(y_i)$$
- $\{x^k, y^k\}$ sequence of **weak** Pareto points for $\{\tilde{P}^{\lambda_k}\}$ with $|x^k| = y^k$.

Theorem:

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- Let $\{\tilde{P}^{\lambda_k}\}$ such that $\forall k$
$$f_m(y) = \sum_{i=1}^{n} f^{\lambda_k}(y_i)$$
- $\{x^k, y^k\}$ sequence of **weak** Pareto points for $\{\tilde{P}^{\lambda_k}\}$ with $|x^k| = y^k$.

Then:

- $\{x^k, y^k\}$ admits limit points
- every limit point is a **weak** Pareto point for the original problem P

Why only weak Pareto? A counterexample...

$$\begin{aligned} \min_{x} \quad & -x, ||x||_0 \\ \text{s.t.} \quad & 0 \leq x \leq 1 \end{aligned}$$

$$\begin{aligned} \min_{x} \quad & -x, 1 - e^{-\lambda_k x} \\ \text{s.t.} \quad & 0 \leq x \leq 1 \end{aligned}$$

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- Pareto points for the original problem: $\{x = 0, x = 1\}$.

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- Pareto points for the original problem: $\{x = 0, x = 1\}$.
- $\forall \lambda_k > 0$, $x_k = \frac{1}{2}$ Pareto point for $\tilde{P}_k^{\lambda}$.

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- Pareto points for the original problem: $\{x = 0, x = 1\}$.
- $\forall \lambda_k > 0$, $x_k = \frac{1}{2}$ Pareto point for $\tilde{P}_k^\lambda$.
- $k \to \infty \implies \bar{x} = \frac{1}{2}$, weak Pareto point for the original problem.

Multiobjective Steepest Descent

$$
\begin{aligned}
\min_{x,y,\theta} \quad & \theta \\
& \nabla f_i(\bar{x})^\top (x - \bar{x}) \leq \theta \qquad \forall i = 1, \ldots, m-1, \\
& \nabla f^\lambda(\bar{y})^\top (y - \bar{y}) \leq \theta, \\
& x \in S \\
& -y \leq x \leq y
\end{aligned}
\tag{LP}
$$

- $\theta^* \leq 0 \quad \forall \bar{x}, \bar{y}$
- x^*, y^* Pareto stationary $\iff \theta^* = 0.$

Algorithm 1: `Steepest Descent Algorithm (SDA)`

Data: $x^0, y^0, \tau > 0$

for $k = 0, 1, \ldots$ **do**

  Compute $\theta, \bar{x}, \bar{y}$ (LP)

  if $|\theta| \leq \tau$ **then**

    Set $x^* = x^k, y^* = y^k$.

    return (*stationarity*).

  end

  Compute α_k with a line search along $v^k = [\bar{x} - x^k, \bar{y} - y^k]$.

  Set $[x^{k+1}, y^{k+1}] = [x^k, y^k] + \alpha_k v^k$.

end

Drawback

SDA improves a single point $\Longrightarrow$ no Pareto front approximation

Solution

- Mantain a list of non dominated points
- Exploit feasible descent directions for the single objectives

Algorithm 2: `MOSO`

Data: $Z = \{z^1, \ldots, z^p\}$ non-dominated points, $\tau > 0$

while *stopping criterion not satisfied* **do**
  for *each* $z \in Z$ **do**
    for *each* f_i **do**
      Compute a feasible descent direction d for f_i in z.
      Compute α^i with a line search along d.
      Set $Z = Z \cup (z + \alpha^i d)$.
    end
    Compute v by solving problem (LP).
    Compute α with a line search along v.
    Set $Z = Z \cup (z + \alpha v)$.
  end
  Set $Z = \{z \in Z \mid z \text{ is non-dominated}\}$[1]
end

[1] Here, the dominance is computed with respect to the original ℓ_0 objective function.

Numerical experiments

Sparse Portfolio problems:

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & -\frac{\mu^T x}{x^T Q x}, \|x\|_0 \\ & e^T x = 1, \\ & 0 \leq x \leq u \end{aligned} \tag{1}$$

Datasets[2]

DTS

- DTS1 (12 assets)
- DTS2 (24 assets)
- DTS3 (48 assets)

Fama-French

- FF10 (10 assets)
- FF17 (17 assets)
- FF48 (48 assets)

[2] R.P. Brito, L. N. Vicente. "Efficient cardinality/mean-variance portfolios", 2013.

Implementation details

Logarithmic approximation

2 initializations:

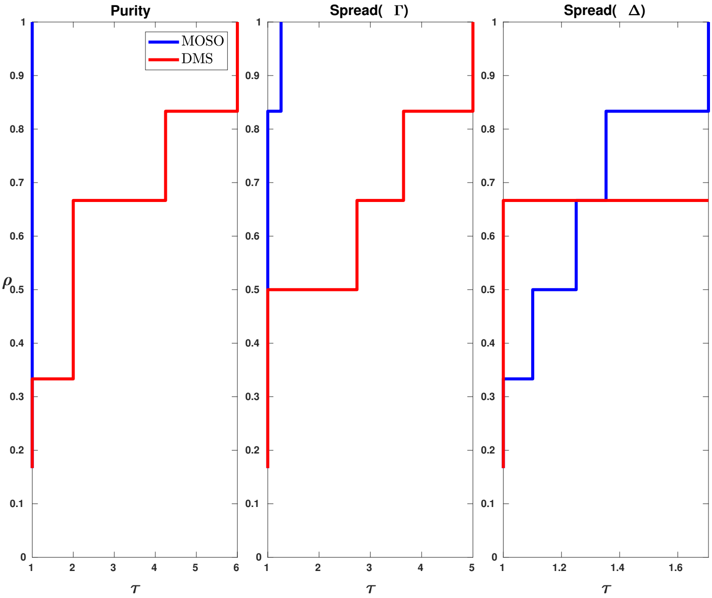
- singleton: $[0, \ldots, 1]$.
- list: 5 random points for each possible cardinality (results for 5 random seeds)

Competitor

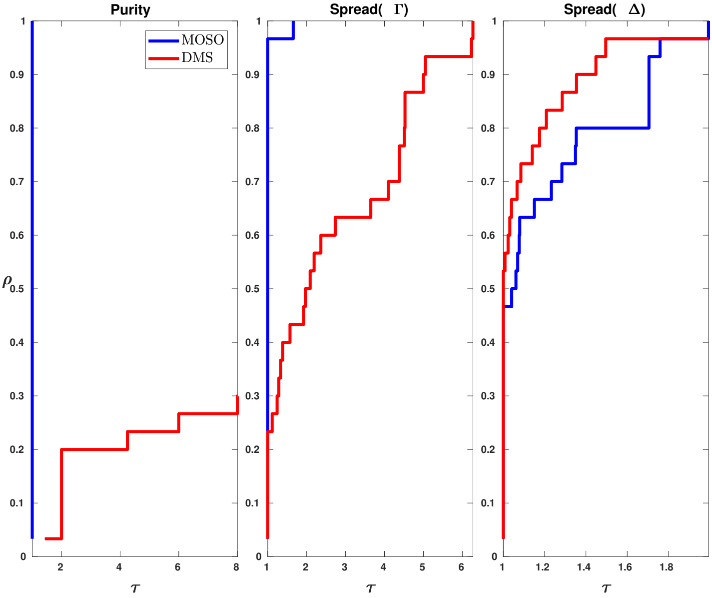
Direct-MultiSearch (DMS)[3]

[3] A.L.Custodio et al. "Direct multisearch for multiobjective optimization", 2011

Singleton initialization



List initialization



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